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Analysis of a neural network model

Christine Fricker¹, Philippe Robert¹, Ellen Saada², Danielle Tibi³.

Abstract.

In this paper we study a stochastic model of a neural network of Purkinje cells proposed by Axelrad et al. [2], [5]. In this model only the inhibitory interaction between the cells is considered. We analyze the stability properties of networks whose graph is complete N-partite. It is shown that if the parameters of the model are below some critical values, then the network converges to unique equilibrium. In this case, the invariant measure of the inhibition states is explicated as well as the distribution of the duration of the interspikes of a given cell. If this stability condition is not satisfied, the state of the network converges to some asymptotic state depending on its initial state. In this case, the set of possible asymptotic states is given.

¹INRIA, domaine de Voluceau, B.P. 105, 78153 Le Chesnay Cédex, France

²CNRS URA 1378, Université de Rouen, B.P. 118, 76134 Cédex, France

³Université de Paris VII, 4 Place Jussieu, 75230 Paris Cédex 05, France

Analyse d'un modèle de réseaux de neurones

Christine Fricker¹, Philippe Robert¹, Ellen Saada², Danielle Tibi³.

Résumé.

Nous étudions un modèle probabiliste d'un réseau de cellules de Purkinje proposé par Axelrad et al. [2], [5]. Dans ce modèle, nous considérons uniquement l'interaction inhibitrice entre les cellules. Si le graphe du réseau est N-partite complet et si les paramètres de celui ci sont inférieurs à certaines constantes, nous montrons que l'état du réseau converge vers un unique état d'équilibre. Dans ce cas, la mesure invariante de l'état du réseau est explicitée, de même que la loi de la durée entre deux excitations consécutives d'une même cellule. Si cette condition de stabilité n'est pas vérifiée, nous donnons les différents états asymptotiques possibles, l'état asymptotique du réseau dépend alors des conditions initiales.

¹INRIA, domaine de Voluceau, B.P. 105, 78153 Le Chesnay Cédex, France

²CNRS URA 1378, Université de Rouen, B.P. 118, 76134 Cédex, France

³Université de Paris VII, 4 Place Jussieu, 75230 Paris Cédex 05, France

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Christine Fricker, Philippe Robert,

INRIA, domaine de Voluceau, B.P. 105, 78153 Le Chesnay Cédex, France

Ellen Saada,

CNRS URA 1378, Université de Rouen, B.P. 118,

76134 Mont Saint Aignan, Cédex, France

Danielle Tibi,

Université de Paris VII, 4 Place Jussieu, 75230 Paris Cédex 05, France

Abstract

In this paper we study a stochastic model of a neural network of Purkinje cells proposed by Axelrad et al. [2], [5]. In this model only the inhibitory interaction between the cells is considered. We analyze the stability properties of networks whose graph is complete N -partite. It is shown that if the parameters of the model are below some critical values, then the network converges to unique equilibrium. In this case, the invariant measure of the inhibition states is explicit as well as the distribution of the duration of the interspikes of a given cell. If this stability condition is not satisfied, the state of the network converges to some asymptotic state depending on its initial state. In this case, the set of possible asymptotic states is given.

Keywords: Neural networks, Markov processes.

1 Introduction

Artificial neural networks generally mimic situations which are quite far from biological reality ; this is due in part to the available experimental tools. The

model we study here is simple enough to enable simulations and a mathematical analysis, and at the same time it is sufficiently close to reality to compare theoretical and experimental results.

Recent studies focus on the cerebellar cortex of the young rat ([4], [2], [3], [5] and the references therein). Its function is to coordinate movements, and it is composed of five types of neurons, synaptically linked and regularly disposed according to a crystalline structure. The main ones, the Purkinje cells (CP), have the only outputs from the cortex. Their study is a step in the (difficult) understanding of the transfer function performed by the cortex. Their activity consists in firing spikes. The Purkinje cells are linked by two types of inhibitory cells, which slow down their activity: The interneural cells, which act on the parallel fibers of CP, and the recurrent collateral cells, which are sprung from the CP axon and act on the somas of neighboring CP. Moreover, the CP's activity is speeded up by climbing fibers, which receive through the granule cells outside inputs to the cortex. During the first postnatal week of the rat, the CP are only linked by inhibitory recurrent collateral cells and excited by climbing fibers. Those last ones can be drug destroyed, and then a record of CP's activity by multi-electrodes ([2]) evaluates the organizing effect of inhibition on the CP's network.

The importance of this effect was enlightened in [3] and [4] by comparisons between networks of isolated CP and networks of CP linked by inhibitory synapses. A first artificial neural network was proposed in [2] to model the activity of CP only submitted to inhibition in a way which kept close to reality, i.e. to the experiences evoked above. Here we study a mathematical representation of this activity introduced in [5], which also only takes into account the inhibition interaction between the Purkinje cells.

From now on Purkinje cells are called neurons, and are assumed to be located at the N vertices of a finite graph, whose edges model the inhibitory cells.

When no connections are present, the different neurons fire spikes according to N independent renewal processes with interarrival distributions G_1, \dots, G_N respectively (where G_i for $1 \leq i \leq N$ is a distribution on \mathbb{R}_+ supposed to have no mass at 0). One natural continuous time process associated with these renewal processes is $(\underline{X}_t)_{t \in \mathbb{R}_+} = (X_t(1), \dots, X_t(N))_{t \in \mathbb{R}_+}$ where $X_t(i)$ is the time interval between t and the next spike of neuron i . The $(X_t(i))_{t \in \mathbb{R}_+}$, $1 \leq i \leq N$, are N independent right continuous Markov processes with arbitrary initial distributions.

We now turn to the case where edges are present and induce an interaction between neighboring (i.e. connected) neurons in the following way : When a spike is sent out by neuron i , it delays by a positive random quantity θ_i the next spikes of the neighbors of i . The process defined in the independent case is then modified by adding random jumps θ_i to the neurons connected to i at the firing times of i . We still get a continuous time Markov process $(X_t(i))_{1 \leq i \leq N, t \in \mathbb{R}_+}$. We call $X_t(i)$ the “inhibition state” of neuron i at time t .

We consider a general class of graphs including the completely connected graphs. We exhibit a critical value of $E(\theta_i)$, $1 \leq i \leq N$, the expectation of θ_i , which depends on the topology. Below this value, the process converges to a unique equilibrium state, i.e. a state in which every neuron stays active. A neuron is said inactive if it stops firing forever, and active otherwise. Thus, in this case the asymptotic behavior of the network does not depend on the initial inhibition states. Above this value, some neurons stay active and others become inactive, we show that there are several possible asymptotic states for the network and that convergence to one of them depends on the initial inhibition states.

In section 2, we give a precise description of our model. In section 3 and 4, we deal with the completely connected network that we study under equilibrium and in the transient case. In section 5, we extend these results to a more general class of graphs, and we conclude in section 6.

2 Description of the model and previous results

We denote by $1 \leq i \leq N$ the vertices of a finite graph and by V_i the set of neighbors of i (i.e. the vertices connected to i by an edge of the graph). The process $(X_t(i), 1 \leq i \leq N)_{t \in \mathbb{R}_+}$ has state space \mathbb{R}_+^N and evolves as follows : From an initial configuration $(X_0(i), 1 \leq i \leq N)$ arbitrarily distributed on $]0, +\infty[^N$, every coordinate decreases linearly in time with slope 1 until the first time T_1 where, for some $1 \leq i_1 \leq N$, the state of neuron i_1 reaches 0. Then i_1 sends out a spike and its state jumps to a value $X_{T_1}(i_1) = E_{i_1}^1$, where $E_{i_1}^1$ is a random variable with distribution G_{i_1} . Simultaneously, i_1 inhibits its neighbors, i.e. increases their states by a random variable $\theta_{i_1}^1$ with distribution F_{i_1} (notice that the same value $\theta_{i_1}^1$ is added to the state of

every $j \in V_{i_1}$). The state of the other neurons remains unchanged. From this configuration $(X_{T_1}(i), 1 \leq i \leq N)$, we iterate this procedure until time T_2 of the second firing, where we make the next jumps (these jumps are distributed according to F_{i_2}, G_{i_2}), and so on.

We have thus defined a right-continuous process which is easily seen to be a Markov process. Now, since its evolution between two successive spikes is deterministic, it is enough to study the discrete time process $(X_{T_n}(i), 1 \leq i \leq N)_{n \in \mathbb{N}}$ which we denote by $(X_n(i), 1 \leq i \leq N)_{n \in \mathbb{N}}$ or $(\underline{X}_n)_{n \in \mathbb{N}}$. It is clearly a Markov chain. Its transition probability is, for $\underline{x} = (x_1, \dots, x_N) \in \mathbb{R}_+^N$ and any measurable function f on \mathbb{R}_+^N if $\min_{1 \leq j \leq N} (x_j) = x_i$ (if more than one i achieves this minimum, then the smallest of them is chosen),

$$P(\underline{x}, f) = \int_{\mathbb{R}_+^N} f(\underline{y}) P(\underline{x}, d\underline{y}) = \int \int f(\phi_i(\underline{x}, \theta, u)) dF_i(\theta) dG_i(u),$$

where $\phi_i(\underline{x}, \theta, u) = \underline{y} \in \mathbb{R}_+^N$, with

$$\begin{cases} y_i = u \\ y_j = x_j - x_i + \theta & \text{if } j \in V_i \\ y_k = x_k - x_i & \text{otherwise.} \end{cases}$$

In the sequel we concentrate on the study of this Markov chain. The first results on it were obtained by M. Cottrell in [5]. It was assumed that for $1 \leq i \leq N$, $G_i = G$ was diffuse and F_i was a Dirac measure on some $\theta > 0$. It was proved that the Markov chain $(\underline{X}_n)_{n \in \mathbb{N}}$ is λ -irreducible (where λ stands for the Lebesgue measure on \mathbb{R}_+^N), aperiodic and that, under the condition $\theta < \frac{\int u dG(u)}{|V|}$ (where $|V|$ denotes the cardinality of V_i , supposed to be independent of i), the chain is positive recurrent. Nevertheless, simulations show that positive recurrence still occurs for some values of θ larger than $\frac{\int u dG(u)}{|V|}$ for some topologies. Hence, this is only a lower bound for the critical value we look for.

We first begin by analyzing the behavior of the network in the case of a completely connected graph (i.e. every pair of neurons is linked by an edge of the graph). The necessary and sufficient condition for the stability of the network is established. Under this condition we give the explicit expression of the stationary distribution of $(\underline{X}_n)_{n \in \mathbb{N}}$ and the distribution of the duration of the interspikes of a given neuron.

In the sequel we assume that, for $1 \leq i \leq N$, G_i is the exponential distribution with parameter λ_i , $\lambda_i > 0$, the case of general G 's will be analyzed in [6].

3 The completely connected network under equilibrium

For $1 \leq i \leq N$, we set $\rho_i = \frac{E(\theta_i)}{E(E_i)} = \lambda_i E(\theta_i)$. In this section we prove that, under the condition $\max_{1 \leq i \leq N} \rho_i < 1$, the Markov chain $(\underline{X}_n)_{n \in \mathbb{N}}$ is ergodic, and we compute the equilibrium distribution of the interspikes of a given neuron in that case.

3.1 Determination of the invariant measure

We denote by ν the product measure $\bigotimes_{i=1}^N G_i$.

Theorem 3.1 *If $\max_{1 \leq i \leq N} \rho_i < 1$, the Markov chain $(\underline{X}_n)_{n \in \mathbb{N}}$ admits the invariant probability measure*

$$\mu = \sum_{i=1}^N \pi_i \mu_i$$

where, for $1 \leq i \leq N$,

$$\pi_i = \frac{\lambda_i}{1 - \rho_i} \left(\sum_{j=1}^N \frac{\lambda_j}{1 - \rho_j} \right)^{-1}$$

and μ_i is the distribution in \mathbb{R}_+^N of the random vector $\underline{E} + (W_i + \theta_i)\underline{e}_i$ where

- . $\underline{E} = (E_1, \dots, E_N)$ has distribution ν ,
- . θ_i has distribution F_i ,
- . $\underline{E}, W_i, \theta_i$ are independent,
- . the i^{th} coordinate of $\underline{e}_i \in \mathbb{R}_+^N$ is equal to 0, the others are equal to 1,

. W_i has the same distribution as the stationary waiting time of an $M/G/1$ queue with arrival rate λ_i and service distribution F_i , i.e.

$$W_i \stackrel{(d)}{=} \max(W_i + \theta_i - E_i, 0), \quad (1)$$

where $\stackrel{(d)}{=}$ stands for equality of distributions. Its Laplace transform is given by

$$\tilde{W}_i(\xi) = \frac{1 - \rho_i}{1 - \lambda_i \frac{1 - \tilde{F}_i(\xi)}{\xi}}, \quad \xi \in \mathbb{R}_+, \quad (2)$$

denoting by \tilde{D} the Laplace transform of a distribution D .

Proof: In order to prove that $\mu P = \mu$, we prove the equality of the Laplace transforms of these quantities. We fix $1 \leq i \leq N$, and we compute $\widetilde{\mu_i P}$, i.e. the Laplace transform of the distribution of \underline{X}_1 if the initial distribution of \underline{X}_0 is μ_i . Thus the initial inhibition state is $X_0(i) = E_i$ for neuron i and $X_0(j) = E_j + W_i + \theta_i$ for $j \neq i$, where W_i, θ_i, E_k , $1 \leq k \leq N$ are independent. The neuron with the smallest inhibition state will fire first. There are two possibilities:

- . If $E_i < W_i + \theta_i$, neuron i fires again and $X_1(i) = E'_i$, $X_1(j) = E_j + (W_i + \theta_i - E_i) + \theta'_i$ for $j \neq i$, where E'_i and θ'_i are other independent random variables with respective distributions G_i and F_i .
- . If $E_i \geq W_i + \theta_i$, $X_0(j) \geq W_i + \theta_i$ for every $1 \leq j \leq N$. Due to the deterministic evolution of $(X_t)_{t \in \mathbb{R}_+}$ between spikes, $P(\underline{x}, \cdot) = P(\underline{y}, \cdot)$ if $y_j = x_j - a$ ($1 \leq j \leq N$) with $a \leq \min_{1 \leq j \leq N} x_j$. Therefore \underline{X}_1 has the same distribution as if the initial state were \underline{X}'_0 with $X'_0(j) = E_j$ for $j \neq i$ and $X'_0(i) = E_i - (W_i + \theta_i)$. Because of the usual properties of exponential distributions, the distribution of \underline{X}'_0 conditioned on $\{E_i \geq W_i + \theta_i\}$ is simply ν .

Thus, for $\underline{\xi} = (\xi_1, \dots, \xi_N) \in \mathbb{R}_+^N$,

$$\begin{aligned} \widetilde{\mu_i P}(\underline{\xi}) &= E \left(1_{\{E_i < W_i + \theta_i\}} e^{-\xi_i E'_i - \sum_{j \neq i} \xi_j (E_j + (W_i + \theta_i - E_i) + \theta'_i)} \right) \\ &\quad + P(E_i \geq W_i + \theta_i) \widetilde{\nu P}(\underline{\xi}). \end{aligned} \quad (3)$$

We are left with the computation of $\widetilde{\nu P}(\underline{\xi})$. Here the initial state \underline{X}_0 is \underline{E} , hence if $E_j = \min_{1 \leq k \leq N} \{E_k\}$, $X_1(j) = E'_j$ and $X_1(k) = E_k - E_j + \theta'_j$ for $k \neq j$. We obtain

$$\widetilde{\nu P}(\underline{\xi}) = \sum_{j=1}^N \frac{\lambda_j}{\sum_{k=1}^N \lambda_k} \tilde{\mu}'_j(\underline{\xi}),$$

where μ'_j is the distribution of the random vector $\underline{E} + \theta'_j e_j$. Because of (1) we have $P(W_i + \theta_i \leq E_i) = P(W_i = 0)$ and this quantity is, according to (2) $1 - \rho_i$. Using the equality of the Laplace transforms of the members of (1), we can rewrite equation (3) as

$$\widetilde{\mu_i P} = \tilde{\mu}_i - (1 - \rho_i) \tilde{\mu}'_i + (1 - \rho_i) \sum_{j=1}^N \frac{\lambda_j}{\sum_{k=1}^N \lambda_k} \tilde{\mu}'_j$$

At this point it is clear that a sufficient condition for the stationarity of $\mu = \sum_{i=1}^N \pi_i \mu_i$ is that π_1, \dots, π_N satisfy

$$\sum_{i=1}^N \pi_i (1 - \rho_i) \left(-(\sum_{k \neq i} \lambda_k) \tilde{\mu}'_i + \sum_{j \neq i} \lambda_j \tilde{\mu}'_j \right) = 0$$

or that for every i , $1 \leq i \leq N$,

$$\pi_i (1 - \rho_i) \sum_{k \neq i} \lambda_k = \lambda_i \sum_{j \neq i} \pi_j (1 - \rho_j).$$

Setting $a_i = \pi_i (1 - \rho_i)$ we get

$$\left(\sum_{k=1}^N \lambda_k \right) \frac{a_i}{\lambda_i} = \sum_{j=1}^N a_j$$

thus $a_i = C \lambda_i$ with $C = \frac{\sum_{j=1}^N a_j}{\sum_{k=1}^N \lambda_k}$.

Finally, $\pi_i = C \frac{\lambda_i}{1 - \rho_i}$, and the normalization relation $\sum_{i=1}^N \pi_i = 1$ (μ is a probability) gives the value of C . Our theorem is proved. ■

3.2 A queueing analogue

The determination of the distribution of the interspikes at equilibrium as well as the study of the instability of the process when $\max_{1 \leq i \leq N} \rho_i > 1$ are strongly related to the notion of busy period (b.p.) for an $M/G/1$ queue.

If neuron i is the first neuron to fire at time $t = 0$, we denote respectively by $(E_i^k)_{k \in \mathbb{N}}$ and $(\theta_i^k)_{k \in \mathbb{N}}$, the sequences of successive firing states of i and inhibitions sent by i . Then, because of the completely connected topology, if for some $n \in \mathbb{N}$, $\sum_{k=1}^l (\theta_i^k - E_i^k) > 0$, for $l = 1, \dots, n$, neuron i will be the only one to fire between 0 and $\sum_{k=1}^n E_i^k$. In this case, if some neuron different from i has inhibition state x at $t = 0$, at time $t = \sum_{k=1}^l E_i^k$ for $l \in \{1, \dots, n\}$, its inhibition state will be $x + \sum_{k=1}^l (\theta_i^k - E_i^k)$ and thus strictly greater than x during that period.

Let us define B_i^1 and τ by

$$B_i^1 = \sup_{n \geq 1} \left\{ \sum_{k=1}^n \theta_i^k \mid \sum_{k=1}^{l-1} (\theta_i^k - E_i^k) > 0, \quad \forall l \leq n \right\} = \sum_{k=1}^{\tau} \theta_i^k.$$

Between 0 and B_i^1 , i will be the only neuron to fire. The successive firing times will be $\sum_{k=1}^{l-1} E_i^k$, $l = 1, \dots, \tau$. At each of these instants, an inhibition distributed as θ_i^l is sent to all the other nodes. Now, the underlying M/G/1 queue is quite clear if we assume that $\sum_{k=1}^l E_i^k$, $l \in \mathbb{N}$, are the arrival instants of the customers who require the services $(\theta_i^l)_{l \in \mathbb{N}}$. In this context, B_i^1 is simply the duration of a busy period starting with one customer. If we denote by $W(t)$ the workload at time t of this queue, for $t \leq B_i^1$, $W(t)$ is easily seen to be the additional inhibition of the other neurons due to i . At $t = B_i^1$, $W(t) = 0$, so that the neurons different from i recover the initial inhibition state they had at time 0 and neuron i has inhibition state $\sum_{k=1}^{\tau} (E_i^k - \theta_i^k)$, which is an exponential variable with parameter λ_i . Then all the inhibition states decrease below their initial value until a neuron fires and generates a new b.p. and so on. Thus, the behavior of the network can be described by a (possibly finite) sequence of b.p. generated by the neurons.

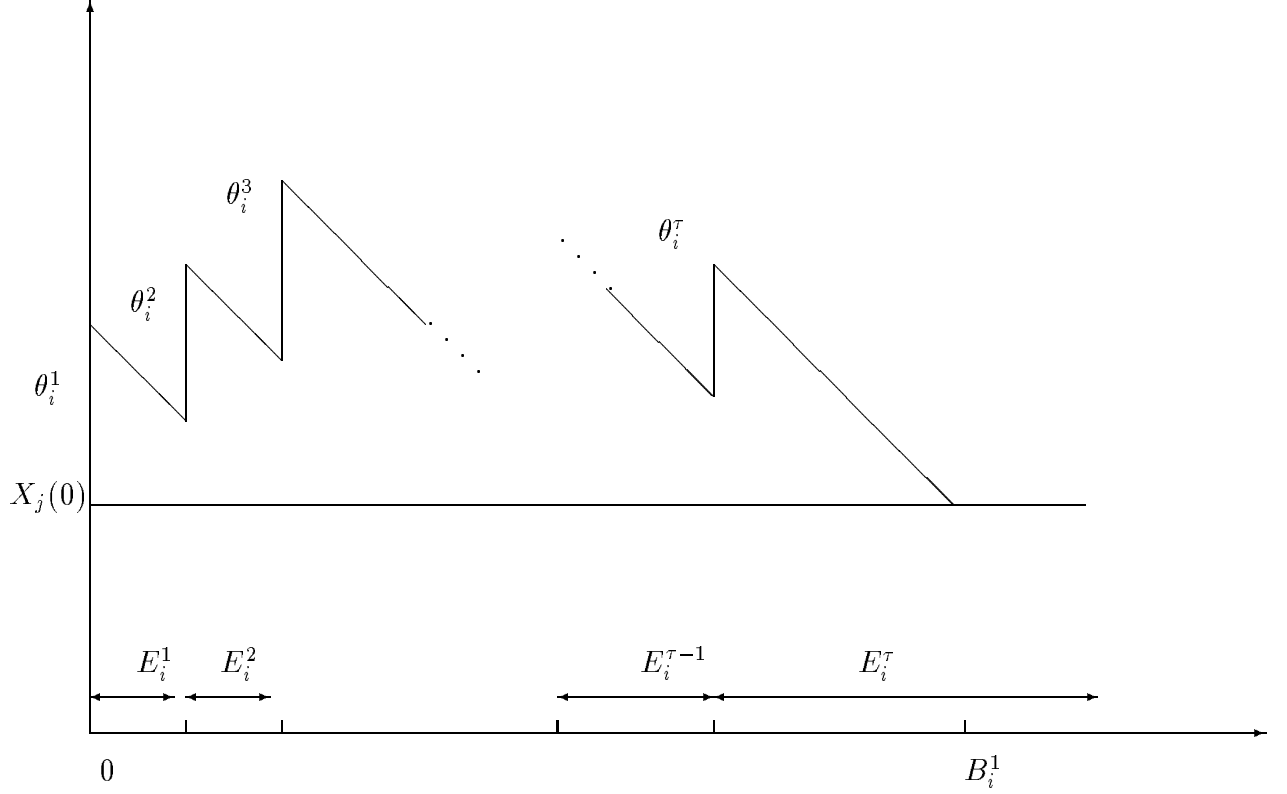


fig. 1 : Inhibition of neuron $j \neq i$ between 0 and B_i^1

We now recall the classical properties of busy periods (see [1]).

Proposition 3.2 *The b.p. B_i is a.s. finite if and only if $\rho_i \leq 1$. In this case the Laplace transform of B_i satisfies the following fixed point equation*

$$\tilde{B}_i(\xi) = \tilde{F}_i(\xi + \lambda_i(1 - \tilde{B}_i(\xi))), \quad \xi \in \mathbb{R}_+. \quad (4)$$

If $\rho_i > 1$, $P(B_i = \infty) = \alpha_i > 0$.

In particular, this implies that if $\rho_i < 1$, then

$$E(B_i) = \frac{E(\theta_i)}{1 - \rho_i}.$$

3.3 The distribution of the interspikes

Proposition 3.3 *Under the stationary regime, the interspike I_i of neuron i has Laplace transform*

$$\tilde{I}_i(\xi) = \frac{\lambda_i}{\lambda_i + \xi} [1 - \tilde{W}_i(\xi + \lambda_i) \tilde{F}_i(\xi + \lambda_i)] + \frac{\lambda_i \tilde{W}_i(\xi + \lambda_i) \tilde{F}_i(\xi + \lambda_i)}{\lambda_i + \xi + \sum_{j \neq i} \lambda_j (1 - \tilde{B}_j(\xi))}, \xi \in \mathbb{R}_+,$$

and expected value

$$E(I_i) = \frac{1}{\lambda_i} + \frac{1 - \rho_i}{\lambda_i} \left(\sum_{j \neq i} \frac{\rho_j}{1 - \rho_j} \right).$$

Proof : From Theorem 3.1 it is clear that μ_i is the stationary distribution μ conditioned on the event that neuron i is firing. Thus, at a spike of i , the inhibition state is E_i for i , and $E_j + W_i + \theta_i$ for $j \neq i$. As in the proof of Theorem 3.1, if neuron i fires again ($E_i < W_i + \theta_i$), the interspike is simply E_i . Otherwise, conditioned on $\{E_i > W_i + \theta_i\}$, at $t = W_i + \theta_i$ the vector of the inhibition states will have distribution ν . In this case the interspike will be $W_i + \theta_i + T_i$ where T_i is the time duration until a spike of i with the initial distribution ν . For $\xi \in \mathbb{R}_+$, we thus get

$$\tilde{I}_i(\xi) = E \left(1_{(E_i < W_i + \theta_i)} e^{-\xi E_i} \right) + E \left(1_{(E_i \geq W_i + \theta_i)} e^{-\xi(W_i + \theta_i)} \right) E(e^{-\xi T_i}),$$

which yields

$$\tilde{I}_i(\xi) = \frac{\lambda_i}{\lambda_i + \xi} \left(1 - \tilde{W}_i(\lambda_i + \xi) \tilde{F}_i(\lambda_i + \xi) \right) + \tilde{W}_i(\lambda_i + \xi) \tilde{F}_i(\lambda_i + \xi) \tilde{T}_i(\xi).$$

To compute $\tilde{T}_i(\xi)$, we consider the first spike starting with distribution ν . Again if i fires, $T_i = E_i = \min_{1 \leq j \leq N} E_j$. Otherwise, a neuron $j \neq i$ will fire and generate a busy period B_j . At the end of B_j , the distribution is ν again, therefore $T_i = E_j + B_j + T'_i$ with $T'_i \stackrel{(d)}{=} T_i$ and the r.v. E_j, B_j, T'_i are independent. We obtain that

$$\tilde{T}_i(\xi) = \frac{\lambda_i}{\xi + \sum_{k=1}^N \lambda_k} + \sum_{j \neq i} \frac{\lambda_j}{\xi + \sum_{k=1}^N \lambda_k} \tilde{B}_j(\xi) \tilde{T}_i(\xi)$$

and

$$\tilde{T}_i(\xi) = \frac{\lambda_i}{\lambda_i + \xi + \sum_{j \neq i} \lambda_j (1 - \tilde{B}_j(\xi))}.$$

Straightforward computations finish the proof. ■

4 Asymptotic behavior of the completely connected network

4.1 Ergodicity

Our aim in this section is to prove the following

Theorem 4.1 *When $\max_{1 \leq j \leq N} \rho_j < 1$, under any initial condition, the Markov chain $(\underline{X}_n)_{n \in \mathbb{N}}$ converges in distribution to the stationary measure μ .*

To show it, we will prove the following proposition, interesting in its own right. It establishes a recurrence property of our Markov process: There exists a non-decreasing sequence of times at which our process has some specified distribution (here ν , the product of exponential distributions). It can be seen as an extension of the classical property of recurrence for Markov processes with discrete state space (see [1] for a discussion of this property).

Proposition 4.2 *If $\max_{1 \leq j \leq N} \rho_j < 1$, then*

- i) almost surely, every neuron fires at least once;*
- ii) there exists a stopping time T , a.s. finite, for the process $(\underline{X}_t)_{t \in \mathbb{R}_+}$ such that \underline{X}_T has distribution ν .*

Proof: Using the queueing analogy, let us denote by $(\underline{Y}_n)_{n \in \mathbb{N}}$ the sequence of inhibition vectors at the beginning of successive busy periods. By our assumption and Proposition 3.2, the busy periods are a.s. finite. Notice that their durations are independent random variables. If neuron i initiates the n -th busy period, then at the end of it, the inhibition state of i will be an exponential random variable independent of the other components of the vector. Therefore, this property will also be true at the beginning of the next busy period, thus $Y_{n+1}(i)$ will be an exponential variable. Now if $Y_{n+1}(i)$ is an exponential r.v. independent of the $Y_{n+1}(j)$, $j \neq i$, then this will also be true for $Y_k(i)$, $k \geq n + 1$.

Let T be the time of the beginning of the first busy period generated by the neuron with the largest initial inhibition state (this neuron fires for the first time after all the others). From what we said above, ii) is a consequence

of i). Notice that, if neuron i never fires, $Y_n(i)$ will be $Y_n(0)$ decreased by the sum of the first $n - 1$ inter busy periods. Because each of these inter busy periods is stochastically greater than an exponential random variable with parameter $\sum_{i=1}^N \lambda_i$ (the minimum of the exponential variables) and that there is an infinite number of b.p., $(Y_n(i))_{n \in \mathbb{N}}$ converges to $-\infty$, giving a contradiction. Hence i) and our proposition are proved. ■

Proof of Theorem 4.1: If $(\underline{X}_n)_{n \in \mathbb{N}}$ is the Markov chain with some initial state $\underline{X}_0 \in \mathbb{R}_+^N$ and $(\underline{X}_n^s)_{n \in \mathbb{N}}$ the chain with initial distribution μ , then $(\underline{X}_n^s)_{n \in \mathbb{N}}$ is a stationary process. Let T and T^s be their respective stopping times of Proposition 4.2. If f is some positive measurable function, then

$$E_\mu(f) = \lim_{n \rightarrow +\infty} E(f(\underline{X}_n^s)) = \lim_{n \rightarrow +\infty} E(f(\underline{X}_n^s/n > T^s)).$$

Using the strong Markov property, we get

$$E_\mu(f) = \lim_{n \rightarrow +\infty} E(f(\underline{X}_n/n > T)) = \lim_{n \rightarrow +\infty} E(f(\underline{X}_n)). \quad \blacksquare$$

Remark

We proved in fact that our Markov chain was Harris ergodic (see [1]). Proposition 4.2 could be refined, so that the sample paths of our chain can be coupled with a stationary version of this chain.

4.2 Instability

In this section we assume that $\rho_1 \leq \rho_2 \leq \dots \leq \rho_N$.

Theorem 4.3 *If there exists $p \in \{0, \dots, N - 1\}$ such that $\rho_1 \leq \rho_2 \leq \dots \leq \rho_p \leq 1 < \rho_{p+1} \leq \dots \leq \rho_N$, then almost surely, there exists $i > p$ such that the inhibition states of all the neurons different from i converge to $+\infty$. Hence, a.s. after a while, $N - 1$ neurons do not fire anymore.*

Proof: By the assumption that $\rho_i > 1$ for $i > p$, according to Proposition 3.2, with probability α_i , the busy period generated by the neuron i will be infinite. Thus if such a neuron fires once, with a positive probability it will be the only one to fire forever, and the inhibition state of the other neurons will be infinite (the workload during the busy period converges to $+\infty$ in this case).

The neurons of indexes less or equal to p generate a.s. finite busy periods, hence with probability one, at least one neuron of index greater than p generates a busy period (cf the proof of Proposition 4.2). Therefore if we assume that there is an infinite number of b.p., there is among them an infinite number with index strictly greater than p . Each of them being infinite with a positive probability (bounded below by $\min_{i>p} P(B_i^1 = +\infty)$), one must be infinite, yielding a contradiction. Hence there is only a finite number of them and the last one is necessarily of index greater than p . Our theorem is proved. ■

Remarks

- a) The sufficient stability condition of [5] is in this case $\max_{1 \leq j \leq N} \rho_j < \frac{1}{N}$ which is quite far from the necessary and sufficient condition. However, this condition can be necessary and sufficient for some topologies (see [6]).
- b) Our results are not completely exhaustive. One might wonder about the behavior of the network under the condition $\rho_1 \leq \rho_2 \leq \dots \leq \rho_p < 1 = \rho_{p+1} = \dots = \rho_N$ for some $p \in \{0, \dots, N\}$. Using classical results of null recurrence, it is easily seen that every neuron fires infinitely often a.s., the interspike of any neuron is finite almost surely but not integrable.

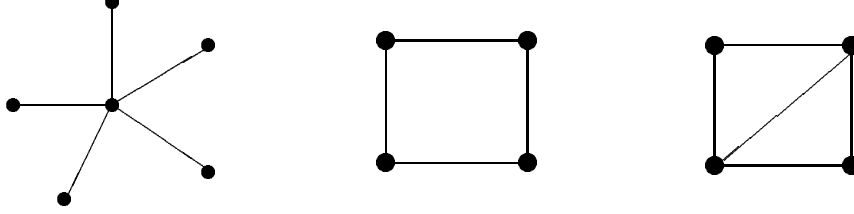
5 The complete N-partite graphs

Definition 5.1 *A graph is N-partite and complete if its vertex set can be decomposed into a partition $\bigcup_1^N H_k$ such that*

1. *two nodes of H_k are not connected for $k \in \{1, \dots, N\}$,*
2. *two nodes from different H 's are connected.*

The completely connected network corresponds to the case where $H_k = \{k\}$ for $k = 1, \dots, N$.

Examples:



We assume that a neuron $j \in H_i$ ($1 \leq i \leq N$) sends out exponentially distributed spikes of parameter λ_i^j and inhibits its neighbors (i.e. neurons of the other blocks $H_j, j \neq i$) with r.v. $(\theta_i^{j,k})_{k \in \mathbb{N}}$ distributed according to F_i^j . We shall denote by C_i the cardinality of H_i . For $1 \leq i \leq N$ we set

$$\rho_i = \sum_{j=1}^{C_i} \lambda_i^j E(\theta_i^j)$$

and we denote by ν the product measure $\bigotimes_{i=1}^N (\bigotimes_{j=1}^{C_i} G_i^j)$. The coordinates of \underline{X}_n are $((X_n^1(j))_{1 \leq j \leq C_1}, \dots, (X_n^N(j))_{1 \leq j \leq C_N})$.

Theorem 5.2 *If $\max_{1 \leq i \leq N} \rho_i < 1$, the Markov chain $(\underline{X}_n)_{n \in \mathbb{N}}$ admits the invariant probability measure*

$$\mu = \sum_{i=1}^N \pi_i \mu_i$$

where for $1 \leq i \leq N$

$$\pi_i = \frac{\hat{\lambda}_i}{1 - \rho_i} \left(\sum_{k=1}^N \frac{\hat{\lambda}_k}{1 - \rho_k} \right)^{-1} \quad \text{with } \hat{\lambda}_i = \sum_{j=1}^{C_i} \lambda_i^j$$

and μ_i is the distribution in $\mathbb{R}_+^{C_1 + \dots + C_N}$ of the random vector $\underline{E} + (W_i + \hat{\theta}_i) \underline{e}_i$ with

. \underline{E} has distribution ν .

. $\hat{\theta}_i$ has distribution $\sum_{j \in H_i} \frac{\lambda_i^j}{\hat{\lambda}_i} F_i^j$.

- . The variables $\underline{E}, W_i, \hat{\theta}_i$ are independent.
- . The coordinates of $\underline{e}_i \in \mathbb{R}_+^{C_1 + \dots + C_N}$ between $C_1 + \dots + C_{i-1} + 1$ and $C_1 + \dots + C_i$ are equal to 0, the others are equal to 1.
- . W_i has the distribution of the stationary waiting time in an $M/G/1$ queue with arrival rate $\hat{\lambda}_i$ and services distributed as $\hat{\theta}_i$.

The proof follows the same lines of the proof of Theorem 3.1, we shall omit it. The same is also true for the following theorem which corresponds to Theorems 4.1 and 4.3 for the completely connected network.

Theorem 5.3 a) If $\max_{1 \leq i \leq N} \rho_i < 1$, then $(\underline{X}_n)_{n \in \mathbb{N}}$ converges in distribution to the invariant measure μ .
b) If $\rho_1 \leq \rho_2 \leq \dots \leq \rho_p \leq 1 < \rho_{p+1} \leq \dots \leq \rho_N$ for some $p \in \{0, \dots, N\}$, almost surely there exists $i > p$ such that the inhibition state of all the neurons of all the H_j with j different from i converge to $+\infty$, and the inhibition states of the neurons of H_i converge to independent exponential variables.

Proof: We think of the blocks $(H_j)_j$ as nodes sending inhibitions distributed as $\hat{\theta}_i$ and restarting with exponentially distributed inhibition states of parameter $\hat{\lambda}_i$. Hence the proof of b) is analogous to the proof of Theorem 4.3. For a), the situation is slightly different. If we look only at the busy periods generated by a block H_i , their durations are no longer independent (remember that H_i was a singleton in the completely connected case). Nevertheless, it is easily seen that after a while, every neuron of H_i will have fired (cf the proof of i) of Proposition 4.2). Using again the memoryless property of exponential variables, after some time, at the beginning of every busy period of H_i , the inhibition states of all the neurons of this block will be independent exponential random variables. This will be also true for the inhibition states of all neurons at the beginning of any busy period. Hence we have the analogue of Proposition 4.2 and we can conclude as in the completely connected case. ■

We finish with a generalization of Proposition 3.3:

Proposition 5.4 Under the assumption that $F_i^j \equiv F_i$ for $j \in H_i$ for all i , the expected value of the interspike of neuron j of block H_i under the stationary

regime is

$$\frac{1}{\lambda_i^j} + \frac{1 - \rho_i}{\lambda_i^j} \left(\sum_{j \neq i} \frac{\rho_j}{1 - \rho_j} \right).$$

Proof : A busy period B_i generated by block i begins at the first spike of a neuron of H_i (after spikes of neurons from other blocks) and it ends when neurons of other blocks recover the state they had at the beginning of B_i . During B_i , neurons of H_i either fire or decrease linearly but they are never inhibited. Moreover, as in the completely connected case, under the initial distribution μ , all the states are exponential at the end of B_i . We denote by \hat{I}_i the interspike of block H_i , i.e. the duration between two consecutive spikes coming from H_i . Using Proposition 3.3, we get

$$E(\hat{I}_i) = \frac{1}{\hat{\lambda}_i} + \frac{1 - \rho_i}{\hat{\lambda}_i} \left(\sum_{j \neq i} \frac{\rho_j}{1 - \rho_j} \right).$$

When a neuron of H_i fires, then the inhibition states of neurons in H_i are exponentially distributed, independent of the past. In particular, the interspikes of H_i are independent. Because of the homogeneity property $F_i^j \equiv F_i$, these interspikes do not depend on the particular neuron j which fires initially. The number of interspikes of H_i until the second firing of j is simply the number of points between 0 and an exponential variable of parameter λ_i^j of an independent Poisson process of parameter $\hat{\lambda}_i - \lambda_i^j$. Straightforward computations finish the proof. ■

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